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NUMERICAL SOLUTION OF RAFTS ON VISCO-ELASTIC MEDIA USING FLEXIBILITY EXPANSIONS

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Abstract—A problem of a raft on a visco-elastic half-space is converted to an equivalent elastic problem by means of a Laplace transformation. A finite element or other suitable numerical technique provides a set of simultaneous equations which depends on the equivalent elastic parameters and which describes the behaviour of the soil-raft system. It is shown that the solution to this set of equations can be expressed in terms of a power series in relative raft flexibility and this solution is then transformed back to obtain a numerical solution to the original problem.

The method is very economical in computing effort since it consists primarily of triangularisation of a matrix, which is much faster than determination of the set of eigenvectors for the matrix. Results for a circular raft are used to illustrate the method and particular attention is paid to the variation in accuracy with the number of terms in the series expansion.

1. INTRODUCTION

The usual method for solving problems of rafts on visco-elastic media is to approximate the visco-elastic equations by means of finite elements and then to apply a forward-marching technique to find the solution at successive time intervals. In the case of simple problems, an analytic solution can be obtained by applying a Laplace transformation to the basic equations. This formally converts the visco-elastic problem to an equivalent elastic problem, and if an explicit solution can be found for the transformed problem, then the solution for the visco-elastic material can be found by inversion of the Laplace transformation.

The numerical method proposed in this paper is closely related to the analytic technique, in that a Laplace transformation of the basic equations of the soil is used to convert the problem to an equivalent elastic problem. Then a finite element or other suitable numerical technique can be used to obtain a set of simultaneous equations which depends on the equivalent elastic parameters and which describes the behaviour of the soil-raft system.

In two previous papers by the present authors[1, 2], an explicit solution to these approximate equations was found in the form of an eigenvector expansion and this explicit solution was transformed back from Laplace transform space to obtain the numerical solution to the visco-elastic problem. The success of this method of solution prompted a search for another method of explicit solution which might be even more efficient in terms of the amount of computation involved.

A power series representation of the solution in terms of relative raft flexibility has been examined and is presented in this paper. Computational economy is achieved because the method consists primarily of triangularisation of the matrix of the governing equation instead of determination of its eigenvectors, yet most of the advantages of the eigenvector method are retained. Both methods allow an explicit solution to the viscoelastic problem when the creep function can be modelled by springs and dashpots and for the case of creep functions which cannot be modelled in this way, allow the solution to be reduced to a series of Volterra integral equations which are solved by a numerical method. Both methods apply to any load pattern, but while the eigenvector method applies to any relative raft stiffness, the present method converges solwly when the range of raft flexibility occurring during the time preiod considered, is very large.

2. BASIC EQUATIONS OF VISCO-ELASTICITY

The most general form of the stress–strain law for a linear visco-elastic material may be written in the form

$$\{\epsilon(t)\} = [C(t)]\{\sigma(0)\} + \int_0^t [C(t-\tau)]\{\dot{\sigma}(\tau)\}\,\mathrm{d}\tau \tag{1}$$

where t = time $\{\epsilon\} = (\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \gamma_{yz}, \gamma_{zx}, \gamma_{xy})^T$ is the vector of strain components $\{\sigma\} = (\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{yz}, \sigma_{zx}, \sigma_{xy})^T$ is the vector of stress components and $[C]$ is the matrix of creep functions, in which $C_{ij}(t)$ is the value of the ith component of strain due to a constant stress $\sigma_j = 1$ applied at $t = 0$. Equation (1) may be considerably simplified by the application of a Laplace transform

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$$

and it then takes the form

$$\{\bar{\sigma}\} = [\bar{D}]\{\bar{\epsilon}\} \tag{2}$$

where $[\bar{D}] = 1/s[\bar{C}]^{-1}$ is the matrix of "transformed" elastic constants and eqn (2) is analogous to Hooke's law for an elastic material. Careful consideration of the equations of compatibility and equilibrium for a visco-elastic material leads to the correspondence principle [3–5] which states that the solution of a visco-elastic problem in terms of Laplace transforms is precisely the same as the solution of an equivalent purely elastic problem.

As an example of the correspondence principle, consider the solution of the Boussinesq problem of a vertical point load P_0 acting on the surface of an isotropic half-space. The solution of the elastic problem is well-known [6] and the vertical surface deflection ω at a distance R from the point of application of the load is

$$\omega(R) = \frac{1}{4\pi}\frac{3K + 4G}{3K + G}\frac{P_0}{GR} \tag{3a}$$

where K and G are the bulk modulus and the shear modulus of the soil. If the correspondence principle is invoked to deduce the solution for a visco-elastic half-space subject to a point load $P(t)$, it is found that

$$\bar{\omega}(R) = \frac{1}{4\pi}\frac{3\bar{K} + 4\bar{G}}{3\bar{K} + \bar{G}}\frac{\bar{P}}{\bar{G}R}. \tag{3b}$$

The variation of ω with time can be found by inverting the Laplace transform and for a constant load the solution will have the form

$$\omega(t) = \frac{P_0}{R} J(t) \tag{4}$$

where $J(t)$ is the inverse transform of $(1/4\pi)(3\bar{K} + 4\bar{G}/3\bar{K} + \bar{G})(1\bar{G}s)$. Provided that $\bar{K}$ and $\bar{G}$ are known, that is provided the volumetric and deviatoric creep functions are known, $J(t)$ can be found by analytical or numerical means. However, of more practical importance is the fact that $J(t)$ can be determined directly from a plate bearing test. In order to show this we consider a circular area of radius a subject to a load distribution of $(P_0/2\pi a^2)$ $(a/\sqrt{[a^2 - r^2]})$ where r is the radial coordinate. Then it follows from the principle of superposition and eqn (4) that the deflection within the circular area is uniform and given by

$$\omega_{\text{rigid}}(t) = \frac{P_0}{2a} J(t). \tag{5}$$

It is interesting to note that eqns (4) and (5) also apply to a cross-anisotropic half-space.

3. ANALYSIS OF THE SOIL-RAFT SYSTEM

It is assumed that the shear stresses developed between the soil and raft are negligible and consequently the tractions at the soil-raft interface are vertical. The condition of compatibility is satisfied by equating the vertical displacements of the soil and raft on each side of the interface.

For the purpose of this numerical approach it is assumed that the soil-raft interface is divided into n elements and that for any element i, the applied loading pressure q_i on the raft and the reaction pressure p_i can be considered to sufficient accuracy as uniform over that element. It is also assumed that the condition of compatibility of deflection at the interface is met if the average deflection of the ith element of the raft ω_{Ri} is equal to the average deflection of the ith element of the soil ω_{si} for all i.

In the case of the soil, it follows from superposition and eqn (3b) that, in the notation of Fig. 1

$$\{\bar{\omega}_s\} = s\bar{J}[I]\{\bar{P}\} \tag{6}$$

where $\{\omega_s\} = (\omega_{s1}, \omega_{s2}, \ldots, \omega_{sn})^T$ = the vector of soil deflections; $\{P\} = (p_1A_1, \ldots p_nA_n)^T$ = the vector of element reaction forces and $[I]$ is the matrix of deflection influence coefficients

$$I_{ij} = \frac{1}{A_iA_j}\int_{(i)}\left\{\int_{(j)}\frac{1}{R_{ij}}\,\mathrm{d}x_j\,\mathrm{d}y_j\right\}\mathrm{d}x_i\,\mathrm{d}y_i.$$

The coefficients I_{ij} may be evaluated analytically for the case of rectangular elements and for more complicated element shapes can be found by numerical integration. When the element forces and deflections are defined as above, the matrix $[I]$ is symmetric and positive definite.

If the raft is considered as a separate body, it will be in equilibrium under the action of the loading stresses q_i and the reaction stresses p_i. However, under the action of these stresses the raft deflections can only be determined relative to an arbitrary rigid body motion. In order to analyse the raft by the finite element method, suppose that just sufficient nodes are fixed to remove this arbitrariness. This leads to a set of equations having the well-known form

$$[K]\{\delta\} = \{F\} \tag{7}$$

where $[K]$ = the stiffness matrix of the raft when its stiffness is S_{ref}; $\{\delta\}$ = the vector of nodal displacements of the raft when its stiffness is S_{ref} and $\{F\}$ = the vector of nodal forces acting on the reft. If the actual stiffness of the raft is S_a, the actual nodal displacements are $S_R\{\delta\}$ where $S_R = S_{ref}/S_a$. The vector of nodal forces can be deduced in a straightforward fashion from the vector of applied element forces $\{Q\}$ and the vector of reaction element forces $\{P\}$ and can always be expressed in the form

$$\{F\} = [M](\{Q\} - \{P\}) \tag{8}$$

and the nodal deflections $S_R \cdot \{\delta\}$ may be determined by solution of eqn (7). The question now arises as to how the nodal deflections should be combined to find the element deflections. One procedure would be merely to average the nodal deflections corresponding to a given element. A more general approach is to equate the expressions for the work done by the nodal forces and nodal deflections to that done by the element forces and element deflections. It then follows that the element deflections without the rigid body motion of the raft are given by

$$\{\omega\} = S_R[M]^T\{\delta\}.$$

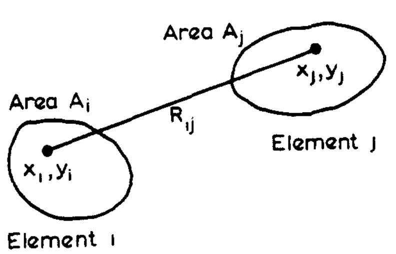

Fig. 1. Notation for eqn (6).

The total element deflections are given by

$$\{\omega\} = S_R[L](\{Q\}-\{P\}) - [A]\{\theta\} \tag{9a}$$

where $[L]=[M]^T[K]^{-1}[M]$; $\{\theta\}$ represents the rigid body motion of the raft and $[A]\{\theta\}$ represents the corresponding motions of the raft elements. This term arises because in deducing the stiffness matrix of the raft, some nodes were fixed to eliminate the rigid body motion from the solution.

An arbitrary set of element forces $\{Q\}-\{P\}$ will in general induce nodal forces at these fixed nodes and to remove such forces it is necessary to observe that the element forces are in equilibrium, hence

$$[A]^T(\{Q\}-\{P\}) = \{0\}. \tag{9b}$$

The behaviour of the complete soil-raft system can be deduced from eqns (6) and (9), so that

$$[\bar{H}]\{\bar{u}\} = \{\bar{v}\} \tag{10a}$$

$$\text{where } [\bar{H}] = \begin{bmatrix} s\bar{J}[I]+S_R[L], & [A] \\ [A]^T, & \{0\} \end{bmatrix} \tag{10b}$$

$$\{\bar{u}\} = \begin{pmatrix} \{\bar{P}\} \\ \{\bar{\theta}\} \end{pmatrix}, \quad \{\bar{v}\} = \begin{pmatrix} S_R[L]\{\bar{Q}\} \\ [A]^T\{\bar{Q}\} \end{pmatrix}.$$

It is possible to deduce the behaviour of a rigid raft by letting $S_a \to \infty$, $S_R \to 0$. It then follows from eqn (10) that

$$\begin{pmatrix} [I],[A] \\ [A]^T_,[0] \end{pmatrix} \begin{pmatrix} \{\bar{P}\} \\ \{\bar{\theta}\}/s\bar{J} \end{pmatrix} = \begin{pmatrix} \{0\} \\ [A]^T\{\bar{Q}\} \end{pmatrix} \tag{11}$$

and thus it can be seen that for constant applied loads, the reaction forces are constant and that

$$\{\omega(t)\} = \frac{J(t)}{J(0)}\{\omega_i\}. \tag{12}$$

4. SOLUTION BY STIFFNESS EXPANSION

Equation (10) may be rewritten in the form

$$[\Phi]\{\bar{\xi}\} = \bar{\lambda}\{\bar{\eta}\} \tag{13}$$

where

$$[\Phi] = \begin{bmatrix} [I]+\bar{\lambda}[L], & \bar{\lambda}[A] \\ \bar{\lambda}[A]^T, & 0 \end{bmatrix}$$

$$\{\bar{\xi}\} = \begin{bmatrix} \{\bar{P}\} \\ k_r\{\bar{\theta}\} \end{bmatrix}, \quad \bar{\eta} = \begin{bmatrix} [L]\{\bar{Q}\} \\ [A]^T\{\bar{Q}\} \end{bmatrix} = \frac{1}{S}\begin{bmatrix} [L]\{Q_0\} \\ [A]^T\{Q_0\} \end{bmatrix}$$

$$\{\bar{\lambda}\} = \frac{\bar{k}_s}{k_r}$$

$$k_r = \frac{S_a}{S_{\text{ref}}} = \frac{1}{S_R} = \text{raft stiffness}$$

$$\bar{k}_s = \frac{1}{s\bar{J}} = \text{Laplace transform of soil stiffness.}$$

Now suppose that $\{\xi\}$ can be expanded as a series in the form

$$\{\xi\} = \frac{1}{s}[\{\xi_0\} + (\bar{\lambda} - \lambda_p)\{\xi_1\} + (\bar{\lambda} - \lambda_p)^2\{\xi_2\} + \ldots]$$

then substituting into eqn (13) we find

$$[\Phi] = [\Phi_0] + (\bar{\lambda} - \lambda_p)[\Phi_1]$$

$$\bar{\lambda}\{\bar{\eta}\} = \frac{1}{s}[\{\eta_0\} + (\bar{\lambda} - \lambda_p)\{\eta_1\}]$$

where

$$[\Phi_0] = \begin{bmatrix} [I] + \lambda_p[L], \lambda_p[A] \\ \lambda_p[A]^T, 0 \end{bmatrix}, [\Phi_1] = \begin{bmatrix} [L], [A] \\ [A]^T, 0 \end{bmatrix}$$

$$\{\eta_0\} = \lambda_p \begin{bmatrix} [L]\{Q_0\} \\ [A]^T\{Q_0\} \end{bmatrix}, \quad \{\eta_1\} = \begin{bmatrix} [L]\{Q_0\} \\ [A]^T\{Q_0\} \end{bmatrix}$$

and that

$$\begin{aligned} &[\Phi_0]\{\xi_0\} + (\bar{\lambda} - \lambda_p)[\Phi_0]\{\xi_1\} + (\bar{\lambda} - \lambda_p)^2[\Phi_0]\{\xi_2\} + \ldots \\ &\quad + (\bar{\lambda} - \lambda_p)[\Phi_1]\{\xi_0\} + (\bar{\lambda} - \lambda_p)^2[\Phi_1]\{\xi_1\} + \ldots \\ &\quad = \{\eta_0\} + (\bar{\lambda} - \lambda_p)\{\eta_1\}. \end{aligned}$$

So that equating coefficients

$$\begin{aligned} [\Phi_0]\{\xi_0\} &= \{\eta_0\} \\ [\Phi_0]\{\xi_1\} &= \{\eta_1\} - [\Phi_1]\{\xi_0\} \\ [\Phi_0]\{\xi_2\} &= -[\Phi_1]\{\xi_1\}. \end{aligned}$$

Clearly $\xi_0, \xi_1, \ldots$ are each just vectors of coefficients whose numerical values can be determined successively once the matrix Φ_o has been triangularised. Then determination of ξ requires only the evaluation of the inverse transforms of

$$\frac{1}{s} \quad \text{denoted by} \quad \phi_0(t)$$

$$\frac{\bar{\lambda} - \lambda_p}{s} \quad \text{denoted by} \quad \phi_1(t)$$

$$\frac{(\bar{\lambda} - \lambda_p)^{2-}}{s} \quad \text{denoted by} \quad \phi_2(t).$$

(This may be done numerically as shown in the Appendix.) Then $\{\xi\} = \{\xi_0\}\phi_0(t) + \{\xi_1\}\phi_1(t) + \{\xi_2\}\phi_2(t) + \ldots$

If the soil displacement matrix $[I]$ is bordered with zeros so as to increase its dimensions to those of matrix Φ in eqn (13), eqn (6) may be replaced by

$$\begin{aligned} \begin{bmatrix} \{\bar{\omega}_s\} \\ 0 \end{bmatrix} &= s\bar{J}\begin{bmatrix} [I], 0 \\ 0, 0 \end{bmatrix}\begin{bmatrix} \{\bar{P}\} \\ k_r\{\bar{\theta}\} \end{bmatrix} \\ &= s\bar{J}\begin{bmatrix} [I], 0 \\ 0, 0 \end{bmatrix}\{\bar{\xi}\} \\ &= s\bar{J}\begin{bmatrix} [I], 0 \\ 0\,0 \end{bmatrix}\frac{1}{s}\left\{\{\xi_0\} + (\bar{\lambda} - \lambda_p)\{\xi_1\} + (\bar{\lambda} - \lambda_p)^2\{\xi_2\} + \ldots\right\} \\ &= \bar{J}\begin{bmatrix} [I], 0 \\ 0, 0 \end{bmatrix}\left\{\{\xi_0\} + (\bar{\lambda} - \lambda_p)\{\xi_1\} + (\bar{\lambda} - \lambda_p)^2\{\xi_2\} + \ldots\right\}. \end{aligned}$$

Then determination of settlements requires only the evaluation of the inverse transforms of

$$\begin{array}{ll} \bar{J} & \text{denoted by } \psi_0 \\ \bar{J}(\bar{\lambda} - \lambda_p) & \text{denoted by } \psi_1 \\ \bar{J}(\bar{\lambda} - \lambda_p)^2 & \text{denoted by } \psi_2\,. \end{array}$$

Then

$$\begin{bmatrix} \omega_0 \\ 0 \end{bmatrix} = \begin{bmatrix} I, 0 \\ 0, 0 \end{bmatrix} \Big\{ \{\xi_0\}\psi_0(t) + \{\xi_1\}\psi_1(t) + \{\xi_2\}\psi_2(t) + \ldots \Big\}.$$

The sets of coefficitnts $\{\xi_i\}$ depend on the pivotal value of flexibility λ_p but not on the initial flexibility λ_0 or the creep function $J(t)$. Different loading patterns merely require use of different right-hand sides for evaluation of a new set of $\{\xi_i\}$ but do not require re-triangularisation of $[\Phi_0]$. The sets of inverse transforms from which the required inverse transforms $\{\phi_i\}$ and $\{\dot{\psi}_i\}$ are determined, depend only on $J(t)$ which can represent any linear viscoelastic model.

5. THEORETICAL RESULTS

An analysis of a uniformly loaded circular raft on a half-space was carried out by dividing the raft into ten annular zones of equal width, under each of which the reaction pressue was assumed uniform. The raft deflections were determined by formal integration of the differential equations for axisymmetrically loaded plates based on thin-plate, small deflection theory. When the reaction distribution at any time had been determined, bending moments were determined by numerical integration.

One of the creep functions adopted was $J(t) = J_f + (J_i - J_f)e^{-t}$ and in this case displacements tend to finite values; $E_i = (1 - \nu_s^2)/J_i$ and $E_f = (1 - \nu_s^2)/J_f$ are respectively the initial and final Young's moduli of the soil. The variation of central element settlement and reaction and central moment with time are shown in Figs. 2–4 for $E_i/E_f = 10$ and for initial relative raft-soil stiffness $K_i = 5.0$, 0.5 and 0.05 where

$$K_i = \frac{E_r(1 - \nu_s^2)}{E_i}\left(\frac{t_r}{a}\right)^3$$

and E_r is the Young's modulus of the raft; ν_s is the Poisson's ratio of the soil; t_r is the raft thickness and a is the raft radius. These three values of K_i correspond to rafts of high, moderate and low stiffness respectively. In the diagrams, q is the loading pressure and p is the reaction pressure for the zone considered and M is the bending moment per unit length.

In each case the results obtained by using 16, 5 and 2 terms in the series expansion have

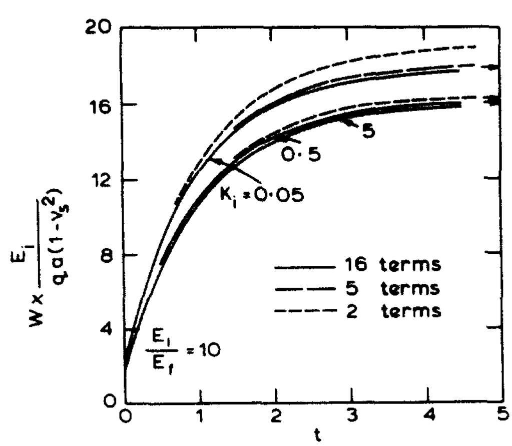

Fig. 2. Central element settlement vs time for uniformly loaded circular raft.

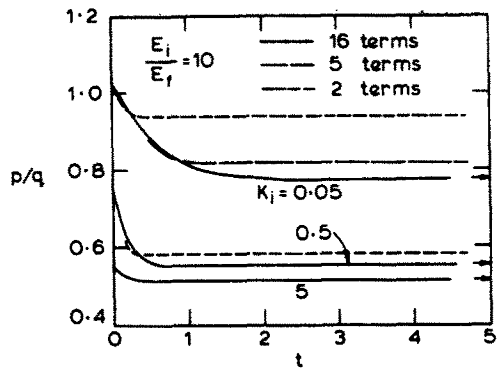

Fig. 3. Central element reaction vs time for uniformly loaded circular raft.

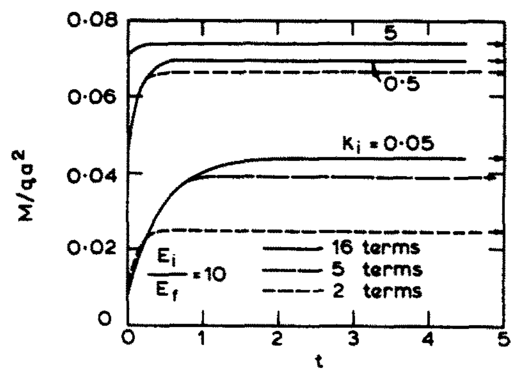

Fig. 4. Central moment vs time for uniformly loaded circular raft.

been presented, except where two of these curves are indistinguishable. The pivotal flexibility (λ_p) was taken as $1/2(\lambda_0 + \lambda_\infty)$ which keeps the value of $\lambda - \lambda_p$ to a minimum.

Further analyses of the uniformly loaded raft were carried out using various values of λ_p, in order to determine the relationship between accuracy of results and the value of $\lambda - \lambda_p$ for various numbers of terms in the expansion. It was found that central moment was the variable whose accuracy was most affected by these changes and that the error in the results obtained was also dependent on the value of λ. Figure 5 shows the relationship between the error in central bending moment and the absolute value of $\lambda - \lambda_p$ at $t = 0$ or $t = \infty$ for 16, 9, 5 and 2 terms in the expansion and for $K = 5.0$ and 0.05. Somewhat larger errors arose for $K = 50$, but

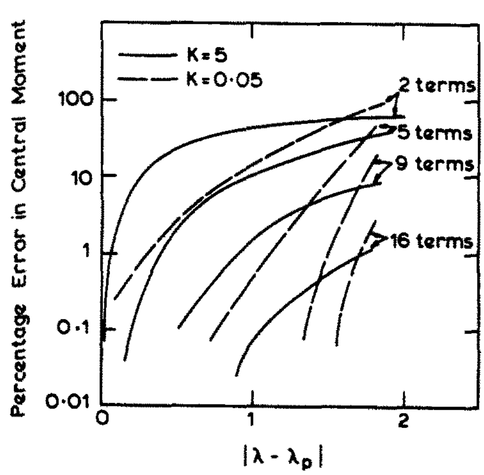

Fig. 5. Variation of error in central moment with relative raft stiffness and number of terms.

for such large stiffnesses (low flexibilities) it will usually be possible to use very small values of $\lambda - \lambda_p$. The accuracy of results appears to be independent of the creep function adopted, since use of $J(t) = J_f + (J_i - J_f)\,e^{-t}$ for $E_i/E_f = 2$ and $J(t) = 1 + \ln(1+t)$, giving $E_i/E_f = \infty$ gave the same values of error in central moment, when the number of terms and the value of $\lambda - \lambda_p$ was the same as used with the original creep function.

For a problem involving 10 "elements" it was found that the computer time required for determination of the eigenvalues and eigenvectors, was more than an order of magnitude greater than the time taken for triangularisation of the matrix as required in the present method. Since the time taken by each of these calculations is roughly proportional to n^3, where n is the number of elements, it seems probable that the ratio of the computer times will remain approximately constant as the number of elements increases, and hence the saving resulting from use of the present method may be considerable.

Solution as an eigenvalue problem requires only one set of inverse transforms, but these must be redetermined for each visco-elastic model and each relative raft-soil stiffness. Whereas with the present method, two sets of inverse transforms are required for each visco-elastic model, and these can be converted into the inverse transforms required for any problem involving that visco-elastic model with little additional computational effort, (see Appendix).

6. CONCLUSIONS

The method of analysis described in this paper can be used to analyse the time settlement behaviour of a raft resting on soil which exhibits any type of linear visco-elastic response, for any load distribution and for any but very large ranges of relative raft-soil flexibility.

The principal advantage of the method is the economy in computational effort, while for any practical range of raft-soil stiffness accuracy of better than 1% can be attained for any variable by use of 16 terms in the expansion.

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APPENDIX

Suppose that $s\bar{J}\bar{f} = \bar{g}$, where the function $g(t)$ is known. This equation may be rewritten in the form

$$[s\bar{J} - J(0)]\bar{f} + J(0)\bar{f} = \bar{g}$$

where, since $s\bar{J} - J(0)$ is the Laplace transform of $J'(t)$, by applying the convolution theorem we find

$$J(0)f(t) + \int_0^t J'(\tau)f(t-\tau)\,d\tau = g(t).$$

This equation enables determination of values of f at various times from the values of the known function $g(t)$.

Writing $t_k = k.\ \Delta t,\ k = 0, 1, \ldots, n;\ \tau_l = l.\ \Delta t,\ l = 0, 1, \ldots, n$ and assuming that the integral can be approximated sufficiently accurately by the trapezoidal rule, we have

$$J(0)f(t_k) + \sum_{l=0}^{k} w_{lk}J'(\tau_l)f(t_k - \tau_l)\Delta t = g(t_k)$$

where $w_{lk} = 1/2$ for $l = 0,\ l = k$
$= 1$ otherwise

This equation may be rewritten in the form

$$J_0 f_k + \sum_{l=0}^{k} w_{lk}J'_l f_{k-l}\Delta t = g_k$$

where $f_k = f(t_k)$ etc., so that

$$(J_0 + 1/2J_0'\Delta t)f_k = g_k - 1/2J_k'f_0\Delta t - \sum_{l=1}^{k-1} J_l'f_{k-l}\Delta t.$$

Hence if $f_0, f_1, \ldots f_{k-1}$ are known, this equation enables f_k to be found, and thus the solution can be marched forward from the initial value $f_0 = g_0/J_0$ to any required value f_n.

Although the form of the actual problem we are concerned with is $\bar{\phi}_i = (\bar{\lambda} - \lambda_p)\bar{\phi}_{i-1}$ where ϕ_0 is known and $\bar{\lambda} = (\bar{k}_s/k_r) = (S_R/s\bar{J})$, for economy in computing it is desirable to produce sets of inverse transforms which are independent of the pivotal point (λ_p) and the initial flexibility (λ_0), and depend only on the creep function $J(t)$. Now since $\bar{\lambda} = (S_R/s\bar{J})$ and $\lambda_0 = (k_s/k_r) = (S_R/J(0))$, $(\bar{\lambda}/\lambda_0) = (J/s\bar{J})$ and clearly the inverse transforms of $(1/s)[(\bar{\lambda} - \lambda_0)/\lambda_0]^i$ are independent of λ_0 and λ_p. For this reason, these inverse transforms were evaluated and subsequently converted to inverse transforms of $(1/s)(\bar{\lambda} - \lambda_p)^i$ for each individual problem.

Thus evaluation consists of solution of $\bar{\psi}_i = \bar{\psi}_{i-1}(\bar{\lambda} - \lambda_0)/\lambda_0$ where ψ_0 is known and successive applications of the method will lead to the functions $\psi_1, \psi_2, \ldots \psi_n$. Substituting for λ and λ_0.

$$\bar{\psi}_i = \frac{J(0)}{s\bar{J}}\bar{\psi}_{i-1} - \bar{\psi}_{i-1}$$

$$s\bar{J}\bar{\psi}_i = J(0)\bar{\psi}_{i-1} - s\bar{J}\bar{\psi}_{i-1}.$$

Now writing $\psi_i = \Psi_i - \chi_i$ where $s\bar{J}\bar{\Psi}_i = J(0)\psi_{i-1}$ then $\chi_i = \bar{\psi}_{i-1}$ and is known from the previous level of solution and determination of Ψ_i is the same as the determination of f given above except for the constant factor $J(0)$.

The conversion of the inverse transforms evaluated, into those required for an individual problem is carried out in the following way.

$$\frac{1}{s}\left(\frac{\bar{\lambda} - \lambda_0}{\lambda_0}\right)^i = \bar{c}_i \text{ has the inverse transform } c_i$$

$$\frac{1}{s}(\bar{\lambda} - \lambda_0) = \bar{a}_i \text{ has the inverse transform } a_i = \lambda_0{}^i c_i$$

$$\frac{1}{s}(\bar{\lambda} - \lambda_p) = \bar{b}_i \text{ has the inverse transform } b_i$$

Now $\bar{b}_i = \frac{1}{s}(\bar{\lambda} - \lambda_0 + \lambda_0 - \lambda_p)^i$

$$\text{so } b_1 = a_1 + (\lambda_0 - \lambda_p)a_0 = \lambda_0 c_1 + (\lambda_0 - \lambda_p)c_0$$

$$b_2 = a_2 + 2(\lambda_0 - \lambda_p)a_1 + (\lambda_0 - \lambda_p)^2 a_0$$

$$= \lambda_0{}^2 c_2 + 2\lambda_0(\lambda_0 - \lambda_p)c_1 + (\lambda_0 - \lambda_p)^2 c_0.$$